Valid restricted randomization for small experiments

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Joint work with Josh Paik (Penn State University)

Introductory example

Suppose that scientists at a horticultural research institute are planning an experiment to compare three varieties of tomato, labelled A, B and C, to see which gives the biggest yield (in weight of fruit per plant). They propose to use a greenhouse which has room for nine tomato plants in a single row. The initial, systematic layout is shown below.

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Devil 2: If you keep doing that, differences between regions will contribute more to the estimate of experimental error than they will to the estimates of differences between varieties, so you may fail to detect genuine differences between varieties.

Angel: Can we use a smaller set of potential layouts with the properties that

(a) we never get a series of 3 adjacent plots with the same variety;

(b) we do not get the bias mentioned by Devil 2?

Some history

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In 1939, Yates proposed the term restricted randomization for any method of randomization that does not include all possible layouts (but preferably avoids both forms of bias).

Terminology: UK, Australia

Plot structure

Inherent nuisance factors

Maybe none

Maybe blocks

Maybe rows and columns

Small units inside large units

Treatment structure

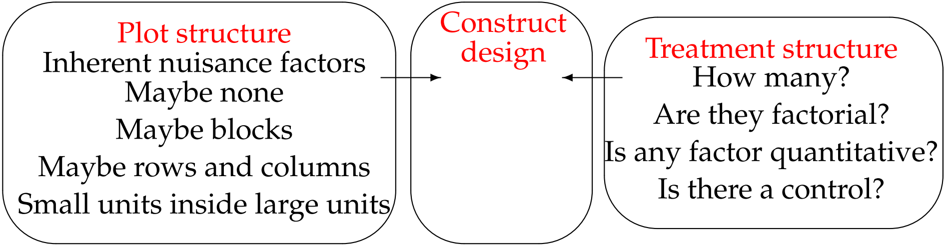
How many?

Are they factorial?

Is any factor quantitative?

Is there a control?

Terminology: UK, Australia



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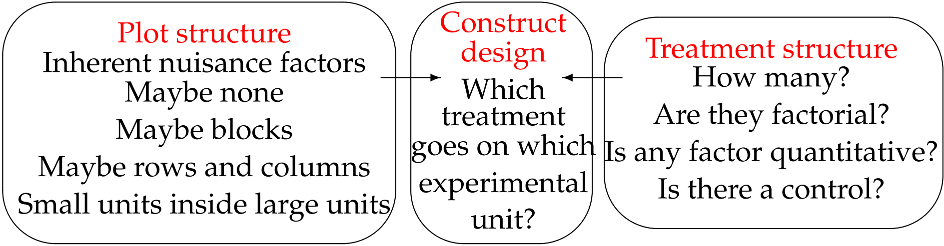

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Which treatment goes on which experimental unit?

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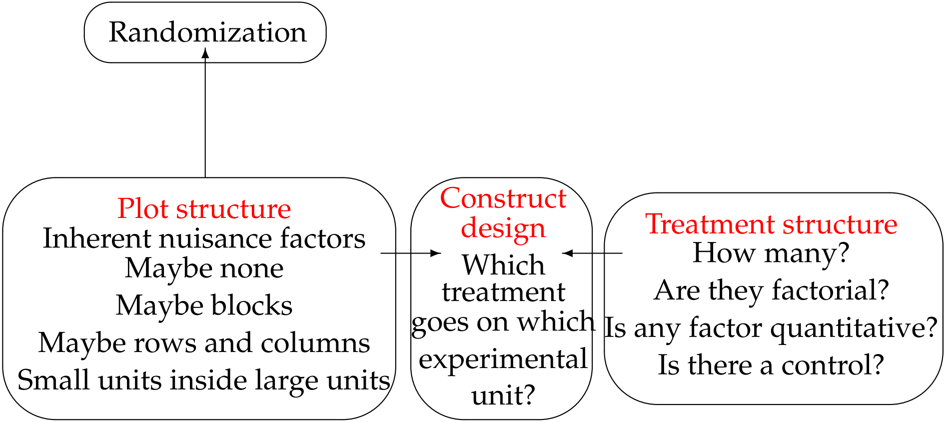

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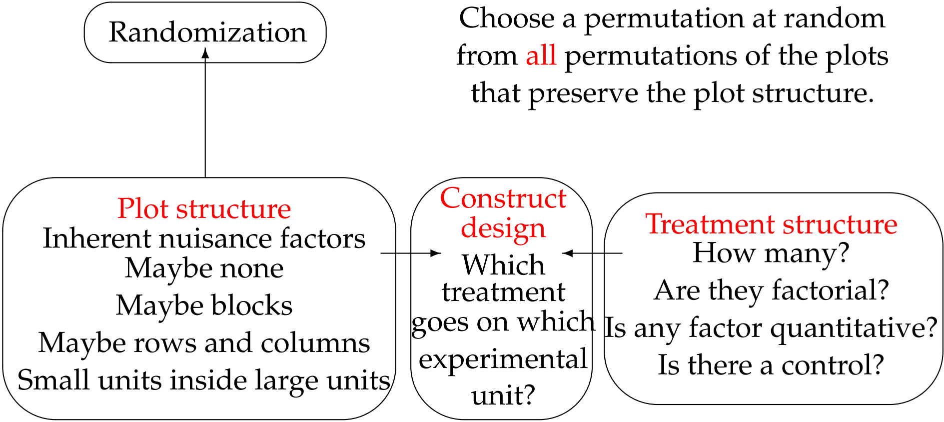

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Which treatment goes on which experimental unit? ←

Treatment structure
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Restricted randomization means using only
a proper subset of the possible layouts.

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They use the term constrained randomization for what I call restricted randomization.

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His approach was close to what is today called causal inference.

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From now on, I will continue to use the term restricted randomization in the sense that Yates did.

Valid randomization

In the context of experiments with a single error term in the analysis of variance,
Fisher and Yates said that a method of randomization should have the property that,
averaged over all possible outcomes of the randomization,
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Random choice of layout for the experiment turns all our statistical notions (such as estimators and mean squares) into random variables.

More technical details

In an unblocked experiment with equal replication, a method of randomization is strongly valid if there are probabilities p_1 and p_2 such that, whenever α and β are distinct plots,

$$P(T(\alpha) = T(\beta) = i) = p_1 \quad \text{for each treatment } i \tag{1}$$

and

$$P(T(\alpha) = i \text{ and } T(\beta) = j) = p_2 \quad \text{whenever } i \neq j. \tag{2}$$

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If a strongly valid method of randomization is used,
then all pairs of distinct plots have the same probability p_2 of contributing to the estimator of the difference between any ordered pair of distinct treatments,
and probability vp_1 of contributing to the mean square for error,
where v is the number of treatments.

Even more technical details

Suppose that there are v treatments, each with replication r, so that the number N of plots is given by $N = vr$. If Equations (1) and (2) are satisfied, then

$$p_1 = \frac{1}{v}\frac{r-1}{N-1} \tag{3}$$

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For a Latin square, there is one such pair for plots in the same row or in the same column, and another such pair for plots which are in different rows and different columns.

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Here we restrict attention to unblocked experiments where the plots form a single line. Denote the design for such an unblocked experiment by Δ.

First strongly valid method, using permutation groups

A group of permutations of the set of N plots is doubly transitive if, whenever α, β, γ and δ are plots with $\alpha \neq \beta$ and $\gamma \neq \delta$, there is a some permutation in the group which takes α to γ and β to δ.

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In 1976–1978, RAB was employed as a post-doc at the Agricultural Research Council Unit of Statistics (in Edinburgh) because her DPhil thesis was about finite permutation groups. This led to a paper on restricted randomization for Latin squares (and other things) in *Biometrika* in 1983.

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After his death in 1971, Youden's widow and the IMS agreed to the publication of Youden's preprint in *Technometrics* in 1972.

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Given such a rectangle, Youden proposed randomizing by choosing one of the rows with equal probability and then randomizing the actual treatments to the letters in that row.

An example from Youden's paper

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There are $3 \times 2 = 6$ columns.

Each row has three letters, each occurring twice.

There is no "very bad" row, such as $AABBCC$.

Each pair of columns have the same letter in precisely one row.

An example from Youden's paper

Suppose that $v = 3$ and $r = 2$.

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Randomize by choosing one of the 5 rows with equal probability, then randomizing the 3 treatments to A, B and C.

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A	*B*	*A*	*C*	*C*	*B*
A	*B*	*B*	*A*	*C*	*C*
A	*B*	*C*	*B*	*A*	*C*
A	*B*	*C*	*C*	*B*	*A*

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We start by finding a resolved balanced incomplete-block design Γ for N treatments in m replicates of blocks of size r, for some value of m.

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Likewise, $P(T(\alpha) \neq T(\beta)) = (m-\lambda)/m$, and this probability is equally split between the $v(v-1)$ ordered pairs of distinct treatments in Δ, and so

$$p_2 = \frac{1}{v(v-1)}\frac{m-\lambda}{m} = \frac{1}{v(v-1)}\frac{(N-r)}{N-1} = \frac{1}{v}\frac{r}{N-1},$$

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As with the first method, the task now is to find a permutation of the columns of the $m \times N$ rectangle such none of the m rows gives a bad pattern.

A solution to the introductory example

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so $v = r = 3$ and $N = 9$.

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Here is a solution with $m = 4$.

The auxiliary design Γ is a balanced square lattice design for 9 treatments.

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so $v = r = 3$ and $N = 9$.
Here is a solution with $m = 4$.

The auxiliary design Γ is a balanced square lattice design for 9 treatments.

1	2	3	4	5	6	7	8	9
A	B	A	C	A	C	C	B	B
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$\{1,3,5\}$	$\{2,8,9\}$	$\{4,6,7\}$
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Moreover, no row has all three occurrences of any letter in either the first or last four columns.

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If $N \geq 26$ then $d \geq 12$, which should give enough power for detecting meaningful treatment differences.

The smallest case has $v = r = 2$.
In this case we cannot avoid the layout $AABB$, and so there is no method of valid restricted randomization.

Replication two, and $v \geq 3$

If $r = 2$, then we can think of the N columns as N vertices of a graph. Each block of Γ contains two of these, so we can think of each block as an edge of the graph.

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Where possible, we ensure that if any letter occurs twice in the first three columns of any row then the last three positions in that row have three different letters.

Valid restricted randomization when $v = 8$ and $r = 2$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
C	*H*	*F*	*D*	*C*	*G*	*E*	*B*	*D*	*A*	*A*	*B*	*H*	*F*	*E*	*G*
B	*H*	*H*	*C*	*D*	*F*	*E*	*A*	*D*	*A*	*B*	*C*	*G*	*F*	*G*	*E*
A	*E*	*H*	*D*	*C*	*H*	*E*	*C*	*F*	*A*	*B*	*B*	*D*	*G*	*F*	*G*
D	*G*	*E*	*C*	*B*	*G*	*D*	*C*	*E*	*A*	*B*	*A*	*H*	*F*	*F*	*H*
D	*H*	*G*	*B*	*A*	*F*	*C*	*C*	*D*	*A*	*B*	*E*	*G*	*E*	*F*	*H*
D	*H*	*G*	*A*	*F*	*E*	*B*	*C*	*C*	*A*	*B*	*E*	*F*	*D*	*H*	*G*
D	*G*	*H*	*G*	*F*	*D*	*A*	*C*	*B*	*A*	*B*	*E*	*E*	*C*	*H*	*F*
D	*F*	*H*	*G*	*F*	*C*	*H*	*C*	*A*	*A*	*B*	*E*	*D*	*B*	*G*	*E*
D	*E*	*G*	*G*	*F*	*B*	*H*	*C*	*H*	*A*	*B*	*E*	*C*	*A*	*F*	*D*
D	*D*	*F*	*G*	*F*	*A*	*G*	*C*	*H*	*A*	*B*	*E*	*B*	*H*	*E*	*C*
D	*C*	*E*	*F*	*F*	*H*	*G*	*C*	*H*	*A*	*B*	*E*	*A*	*G*	*D*	*B*
D	*B*	*D*	*F*	*E*	*G*	*G*	*C*	*H*	*A*	*B*	*E*	*H*	*F*	*C*	*A*
D	*A*	*C*	*F*	*E*	*F*	*E*	*C*	*G*	*A*	*B*	*D*	*H*	*H*	*B*	*G*
C	*H*	*B*	*F*	*E*	*E*	*G*	*C*	*D*	*A*	*B*	*D*	*G*	*H*	*A*	*F*
C	*F*	*A*	*C*	*E*	*G*	*D*	*B*	*F*	*A*	*B*	*D*	*G*	*E*	*H*	*H*

Two treatments, and $r \geq 3$

When $v = 2$ and $r = 3$, then Γ must be a resolved balanced incomplete-block design for 6 treatments in blocks of size 3. The smallest such design consists of all triples of treatments, and so we cannot avoid the layout *AAABBB*.

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For larger even values of r, we started with a normalized Hadamard matrix of order $2r$. This is a $2r \times 2r$ matrix with all entries equal to 1 or -1. All entries in the first row are $+1$. In each other row, half the entries are 1 and half are -1. Every pair of rows are orthogonal to each other.

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We removed the first row to form an $m \times N$ rectangle with $N = 2r$ and $m = N - 1$. We replaced $+1$ by A and -1 by B.

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We removed the first row to form an $m \times N$ rectangle with $N = 2r$ and $m = N - 1$. We replaced $+1$ by A and -1 by B. Then we permuted the columns until the longest subsequence of identical letters in any row was as small as possible. Where possible, we avoided having such sequences at either end of the row.

Valid restricted randomization when $v = 2$ and $r = 6$

1	2	3	4	5	6	7	8	9	10	11	12
A	B	A	A	B	A	B	A	A	B	B	B
A	B	B	B	A	A	B	B	A	B	A	A
A	A	B	A	B	A	A	B	B	B	B	A
A	B	A	B	B	B	A	B	A	A	B	A
A	B	B	B	A	A	A	A	B	A	B	B
A	B	B	A	B	B	B	A	B	A	A	A
A	A	B	B	B	B	A	A	A	B	A	B
A	A	A	B	B	A	B	B	B	A	A	B
A	A	A	B	A	B	B	A	B	B	B	A
A	B	A	A	A	B	A	B	B	B	A	B
A	A	B	A	A	B	B	B	A	A	B	B

Two treatments, and $r \geq 3$ with r odd

For these values, a slightly more complicated construction uses a normalized Hadamard matrix of order $4r$.

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We remove the first two rows, and we remove the $2r$ columns which have entry -1 in the second row.
This gives a $2(2r-1) \times 2r$ rectangle with the right properties.

Two treatments, and $r \geq 3$ with r odd

For these values, a slightly more complicated construction uses a normalized Hadamard matrix of order $4r$.

We remove the first two rows, and we remove the $2r$ columns which have entry -1 in the second row.
This gives a $2(2r-1) \times 2r$ rectangle with the right properties.

As before, we permute columns to avoid long subsequences of identical letters, particularly at either end of a row.

Larger values of v and r

When $v = r \geq 3$ we can use balanced square lattice designs when $v = 3$, $v = 4$ or $v = 5$.

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For other larger values, we have to find a resolved balanced-incomplete block design Γ separately in each case.

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When $v = r \geq 3$ we can use balanced square lattice designs when $v = 3$, $v = 4$ or $v = 5$.

For other larger values, we have to find a resolved balanced-incomplete block design Γ separately in each case. For example, here is a valid restricted randomization scheme for $v = 5$ and $r = 3$.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
A	C	E	B	E	B	D	D	A	E	C	C	D	A	B
A	A	C	B	D	C	D	A	B	B	E	D	E	C	E
B	A	D	B	A	E	E	C	E	C	C	D	B	A	D
E	B	D	B	C	C	B	A	D	E	D	E	C	A	A
D	E	E	B	A	D	C	A	A	C	D	B	E	B	C
A	D	B	B	A	D	E	B	C	D	E	C	C	E	A
C	A	C	B	B	E	C	E	A	D	B	E	D	D	A

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