On a new approach to the analysis of variance for experiments with orthogonal block structure.

II. Experiments in nested block designs

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Abstract

- Main estimation and hypothesis testing procedures are presented for experiments conducted in nested block designs of certain type.

- It is shown that under appropriate randomization these experiments have the convenient orthogonal block structure.

- Due to it, the analysis of experimental data can be performed in a simple way.

- The main advantage of the presented methodology concerns the analysis of variance and related hypothesis testing procedures.

- Under the adopted approach one can reach them directly, not by combining results from the analyses based on stratum submodels.

- Application of the presented theory is illustrated by three examples of real experiments in relevant nested block designs.

The Scheme of Presentation

1. Introduction

2. Randomization-derived mixed model, from which the described methodology follows

3. Theoretical background of the derived analysis

4. Some simplifications of the proposed analytical methods

5. Attention drawn to some modifications resulting from the use of estimated stratum variances

6. Some examples illustrating application of the derived analytical methods

7. Concluding remarks

1. Introduction

The concept of orthogonal block structure, as a desirable property, was originally considered for a wide class of designs by Nelder (1965) and then formalized by Houtman and Speed (1983).

Definition 1.1 (from Section 2.2 in Houtman and Speed, 1983). An experiment is said to have the orthogonal block structure (OBS) if the covariance (dispersion) matrix of the random variables observed on the experimental units (plots), $\boldsymbol{y} = [y_1, y_2, ..., y_n]'$, has a representation of the form

$$\mathrm{D}(\boldsymbol{y}) = \sigma_1^2\boldsymbol{\phi}_1 + \sigma_2^2\boldsymbol{\phi}_2 + \cdots + \sigma_t^2\boldsymbol{\phi}_t,$$

where the $\{\boldsymbol{\phi}_\alpha\}$, $\alpha = 1, 2, ..., t$, are known symmetric, idempotent and pairwise orthogonal matrices, summing to the identity matrix, the last being usually of the form $\boldsymbol{\phi}_t = n^{-1}\mathbf{1}_n\mathbf{1}_n'$.

Experiments having the OBS property can be analyzed in a simple way.

The analysis of variance (ANOVA) can be performed directly, avoiding the classic procedure of first conducting the analyses based on stratum submodels and then combining informations obtained from them, as originally suggested by Yates (1939, 1940) and recently discussed by Kala (2017).

The present paper, as the second in the projected series of publications, is devoted to experiments conducted in nested block (NB) designs.

NB designs are often used in practice. Their statistical properties have been considered in many papers, as reviewed by Bailey (1999).

Of special interest are those NB designs which induce the OBS property as indicated in Caliński and Kageyama (2000, Lemma 5.4.1).

The purpose of the present paper is to show how the OBS property of an experiment in NB design gives the possibility of performing the analysis of experimental data with a comparatively simple methodology, similarly as in the first paper of the present series (Caliński and Siatkowski, 2017).

2. A randomization-derived model

Consider an experiment in an NB design with v treatments (varieties) in $b = ab_0$ blocks, each of k units (plots), grouped into a superblocks composed of b_0 blocks each. Such an NB design is said to induce the OBS property.

Suppose that independent randomizations of superblocks, of blocks within the superblocks and of plots within the blocks have been implemented in the experiment. The randomization-derived model can then be written as

$$\boldsymbol{y} = \boldsymbol{X}_1\boldsymbol{\tau} + \boldsymbol{X}_A\boldsymbol{\alpha} + \boldsymbol{X}_B\boldsymbol{\beta} + \boldsymbol{\eta} + \boldsymbol{e}, \tag{1}$$

where $\boldsymbol{y}$ is an $n \times 1$ vector of data observed on $n = ab_0k$ plots, $\boldsymbol{X}_1$, $\boldsymbol{X}_A$, $\boldsymbol{X}_B$ are the known design matrices, and where $\boldsymbol{\tau} = [\tau_1, \tau_2, ..., \tau_v]'$ represents the unobservable treatment parameters, whereas $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, ..., \alpha_a]'$ stands for the superblock random effects, $\boldsymbol{\beta} = [\boldsymbol{\beta}_1', \boldsymbol{\beta}_2', ..., \boldsymbol{\beta}_a']'$, with $\boldsymbol{\beta}_h = [\beta_{1(h)}, \beta_{2(h)}, ..., \beta_{b_0(h)}]'$, for the block random effects, while the $n \times 1$ vectors $\boldsymbol{\eta}$ and $\boldsymbol{e}$ stand for the unit error and technical error random variables.

The whole block design, denoted by $\mathcal{D}^*$, can be described by the $v \times b$ incidence matrix

$$\boldsymbol{N} = \boldsymbol{X}_1'\boldsymbol{X}_B = [\boldsymbol{N}_1 : \boldsymbol{N}_2 : \cdots : \boldsymbol{N}_a],$$

with $\boldsymbol{N}_h = \boldsymbol{X}_{1h}'\boldsymbol{X}_{Bh}$ describing the hth component design, denoted by $\mathcal{D}_h$, where $\boldsymbol{N}_h'\mathbf{1}_v = k\mathbf{1}_{b_0}$ and $\boldsymbol{N}_h\mathbf{1}_{b_0} = \boldsymbol{r}_h$, the vector of treatment replications in $\mathcal{D}_h, h = 1, 2, ..., a$. Furthermore, note that the design, denoted by $\mathcal{D}$, by which the v treatments are assigned to the a superblocks is described by the $v \times a$ incidence matrix

$$\boldsymbol{M} = \boldsymbol{X}_1'\boldsymbol{X}_A = [\boldsymbol{r}_1 : \boldsymbol{r}_2 : \cdots : \boldsymbol{r}_a].$$

Because both $\mathcal{D}^*$ and $\mathcal{D}$ are proper, an experiment in such NB design has the OBS property. This allows the model to be resolved into four simple stratum submodels. Using Nelder's (1965) notation, this stratification can be represented by

$$\text{Units (plots)} \rightarrow \text{Blocks} \rightarrow \text{Superblocks} \rightarrow \text{Total area.}$$

Thus, the observed vector $\boldsymbol{y}$ can be decomposed as

$$\boldsymbol{y} = \boldsymbol{y}_1 + \boldsymbol{y}_2 + \boldsymbol{y}_3 + \boldsymbol{y}_4, \quad \boldsymbol{y}_1 = \boldsymbol{\phi}_1\boldsymbol{y}, \;\; \boldsymbol{y}_2 = \boldsymbol{\phi}_2\boldsymbol{y}, \;\; \boldsymbol{y}_3 = \boldsymbol{\phi}_3\boldsymbol{y}, \;\; \boldsymbol{y}_4 = \boldsymbol{\phi}_4\boldsymbol{y},$$

which allows the expectation vector and the covariance (dispersion) matrix of $\boldsymbol{y}$ to be written as

$$\mathrm{E}(\boldsymbol{y}) \;=\; \boldsymbol{\phi}_1\boldsymbol{X}_1\boldsymbol{\tau} + \boldsymbol{\phi}_2\boldsymbol{X}_1\boldsymbol{\tau} + \boldsymbol{\phi}_3\boldsymbol{X}_1\boldsymbol{\tau} + \boldsymbol{\phi}_4\boldsymbol{X}_1\boldsymbol{\tau} = \boldsymbol{X}_1\boldsymbol{\tau}, \tag{2}$$

$$\mathrm{D}(\boldsymbol{y}) \equiv \boldsymbol{V} \;=\; \sigma_1^2\boldsymbol{\phi}_1 + \sigma_2^2\boldsymbol{\phi}_2 + \sigma_3^2\boldsymbol{\phi}_3 + \sigma_4^2\boldsymbol{\phi}_4, \tag{3}$$

where the matrices

$$\boldsymbol{\phi}_1 = \boldsymbol{I}_n - k^{-1}\boldsymbol{X}_B\boldsymbol{X}'_B, \;\; \boldsymbol{\phi}_2 = k^{-1}\boldsymbol{X}_B\boldsymbol{X}'_B - n_0^{-1}\boldsymbol{X}_A\boldsymbol{X}'_A,$$

$$\boldsymbol{\phi}_3 = n_0^{-1}\boldsymbol{X}_A\boldsymbol{X}'_A - n^{-1}\mathbf{1}_n\mathbf{1}'_n \;\text{ and }\; \boldsymbol{\phi}_4 = n^{-1}\mathbf{1}_n\mathbf{1}'_n$$

are symmetric, idempotent and pairwise orthogonal, summing to the identity matrix, and $\sigma_1^2, \sigma_2^2, \sigma_3^2$ and σ_4^2 represent the relevant unknown stratum variances (defined as in Caliński and Kageyama, 2000, Section 5.4).

3. Theoretical background of the analysis

When analyzing data from an experiment modelled by (1), a variety trial in particular, attention is usually paid to the parameters $\boldsymbol{\tau} = [\tau_1, \tau_2, ..., \tau_v]'$, or rather the treatment (variety) main effects

$$(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\boldsymbol{\tau} = [\tau_1 - \tau_., \tau_2 - \tau_., ..., \tau_v - \tau_.]', \quad \text{where} \quad \tau_. = n^{-1}\sum_{i=1}^{v}(r_i\tau_i),$$

and also their linear functions. In connection with this, first note that, taking the orthogonal ($\boldsymbol{V}^{-1}$-orthogonal) projector

$$\boldsymbol{P}_{X_1(V^{-1})} = \boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}, \tag{4}$$

one can decompose the vector $\boldsymbol{y}$ as

$$\boldsymbol{y} = \boldsymbol{P}_{X_1(V^{-1})}\boldsymbol{y} + (\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}. \tag{5}$$

Under the model (1) the first term of the partition in (5) provides the BLUE of $\boldsymbol{X}_1\boldsymbol{\tau}$ in (2), as

$$\widehat{\boldsymbol{X}_1\boldsymbol{\tau}} = \boldsymbol{P}_{X_1(V^{-1})}\boldsymbol{y}, \tag{6}$$

as it follows from Rao (1974, Theorem 3.2). The second term in (5) can be seen as the residual vector, giving the residual sum of squares

$$\begin{aligned}\|(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2_{V^{-1}} =& \boldsymbol{y}'(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})'\boldsymbol{V}^{-1}(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y} \\ =& \boldsymbol{y}'[\boldsymbol{V}^{-1} - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}]\boldsymbol{y} \\ =& \boldsymbol{y}'\boldsymbol{V}^{-1}(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y},\end{aligned} \tag{7}$$

with the residual degrees of freedom (d.f.) given by $\text{rank}(\boldsymbol{V} : \boldsymbol{X}_1) - \text{rank}(\boldsymbol{X}_1) = n - v$. See Rao (1974, Theorem 3.4) and formula (3.17) there. Note that, when using the projector (4) in the considered applications, the variance σ_4^2 in the involved matrix $\boldsymbol{V}$ can be replaced by 1.

It will be interesting to note that, as $\boldsymbol{\tau} = \boldsymbol{r}^{-\delta}\boldsymbol{X}_1'\boldsymbol{X}_1\boldsymbol{\tau}$, the BLUE of $\boldsymbol{\tau}$ can be obtained, on account of the formulae (4) and (6), as

$$\hat{\boldsymbol{\tau}} = (\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{y}. \tag{8}$$

Its covariance (dispersion) matrix gets the form

$$\begin{aligned}\mathrm{D}(\hat{\boldsymbol{\tau}}) &= (\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}\mathrm{D}(\boldsymbol{y})\boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1} \\ &= (\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}. \end{aligned} \tag{9}$$

These results can be checked by referring to Theorem 3.1 in Rao (1971).

With these results the concept of testing the hypothesis

$$H_0 : (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\boldsymbol{\tau} = \mathbf{0}, \tag{10}$$

can be considered. For this note that the BLUE of $\boldsymbol{\tau}_* = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\boldsymbol{\tau}$ is $\hat{\boldsymbol{\tau}}_* = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\hat{\boldsymbol{\tau}}$. Its dispersion matrix is

$$\mathrm{D}(\hat{\boldsymbol{\tau}}_*) = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v'), \quad \text{rank } v-1. \tag{11}$$

It appears that

$$\mathrm{D}(\hat{\boldsymbol{\tau}}_*)[\mathrm{D}(\hat{\boldsymbol{\tau}}_*)]^- \hat{\boldsymbol{\tau}}_* = \hat{\boldsymbol{\tau}}_* \tag{12}$$

indicates that H_0 in (10) is consistent; see formula (3.2.8) in Rao (1971).

Assuming now that $\boldsymbol{y} \sim N_n(\boldsymbol{X}_1\boldsymbol{\tau}, \boldsymbol{V})$ and, hence, that $\hat{\boldsymbol{\tau}}_* \sim N_v[\boldsymbol{\tau}_*, \mathrm{D}(\hat{\boldsymbol{\tau}}_*)]$, where $\boldsymbol{\tau}_*$ is as defined above, and $\mathrm{D}(\hat{\boldsymbol{\tau}}_*)$ is as in (11), one can test the hypothesis H_0 using the statistic

$$F = \frac{n-v}{v-1}\,\frac{\mathrm{SS}_V}{\mathrm{SS}_R} = \frac{n-v}{v-1}\,\frac{\hat{\boldsymbol{\tau}}_*'\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1\hat{\boldsymbol{\tau}}_*}{\boldsymbol{y}'\boldsymbol{V}^{-1}(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}}, \tag{13}$$

as it follows from Theorem 3.2 in Rao (1971). Note, however, that

$$\mathrm{SS}_V = \boldsymbol{y}'\boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v')\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{y}, \tag{14}$$

$$\mathrm{SS}_R = \boldsymbol{y}'[\boldsymbol{V}^{-1} - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}]\boldsymbol{y}. \tag{15}$$

Referring now to Theorems 9.2.1 and 9.4.1 in Rao and Mitra (1971), one can show that, independently,

$$\mathrm{SS}_V \sim \chi^2(v-1, \delta), \quad \text{with} \quad \delta = \boldsymbol{\tau}_*' \boldsymbol{X}_1' \boldsymbol{V}^{-1} \boldsymbol{X}_1 \boldsymbol{\tau}_*, \tag{16}$$

$$\mathrm{SS}_R \sim \chi^2(n-v, 0). \tag{17}$$

Evidently, the distribution in (16) is central if H_0 is true, whereas that in (17) is central, whether H_0 is true or not. These results imply that the statistic (13) has a noncentral F distribution with $v-1$ and $n-v$ d.f., and with the noncentrality parameter δ as in (16). Thus, the distribution is central if H_0 is true.

It should be noted that the above estimation and hypothesis testing procedures are applicable directly if the stratum variances σ_1^2, σ_2^2, σ_3^2 and σ_4^2 are known. In practice they have to be estimated.

It will be helpful to return to formula (7), writing it as

$$\begin{aligned}\|(\boldsymbol{I}_n-\boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2_{V^{-1}} =& \boldsymbol{y}'(\boldsymbol{I}_n-\boldsymbol{P}_{X_1(V^{-1})})'\boldsymbol{V}^{-1}(\boldsymbol{I}_n-\boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\\ =& \sigma_1^{-2}\boldsymbol{y}'(\boldsymbol{I}_n-\boldsymbol{P}_{X_1(V^{-1})})'\boldsymbol{\phi}_1(\boldsymbol{I}_n-\boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\\ &+\sigma_2^{-2}\boldsymbol{y}'(\boldsymbol{I}_n-\boldsymbol{P}_{X_1(V^{-1})})'\boldsymbol{\phi}_2(\boldsymbol{I}_n-\boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\\ &+\sigma_3^{-2}\boldsymbol{y}'(\boldsymbol{I}_n-\boldsymbol{P}_{X_1(V^{-1})})'\boldsymbol{\phi}_3(\boldsymbol{I}_n-\boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}, \quad (18)\end{aligned}$$

which follows from the form of $\mathrm{D}(\boldsymbol{y}) \equiv \boldsymbol{V}$, given in (3). This also implies, on account of the relation $\boldsymbol{\phi}_4 = n^{-1}\mathbf{1}_n\mathbf{1}'_n = n^{-1}\mathbf{1}_n\mathbf{1}'_v\boldsymbol{X}'_1$, that

$$\boldsymbol{\phi}_4(\boldsymbol{I}_n-\boldsymbol{P}_{X_1(V^{-1})}) = \sigma_4^2\boldsymbol{\phi}_4\boldsymbol{V}^{-1}(\boldsymbol{I}_n-\boldsymbol{P}_{X_1(V^{-1})}) = \mathbf{O}.$$

Now, from (18), one can write

$$\begin{aligned}\mathrm{E}\{\|(\boldsymbol{I}_n-\boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2_{V^{-1}}\} &= \sigma_1^{-2}\mathrm{E}\{\|\boldsymbol{\phi}_1(\boldsymbol{I}_n-\boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2\}\\ &\quad+\sigma_2^{-2}\mathrm{E}\{\|\boldsymbol{\phi}_2(\boldsymbol{I}_n-\boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2\}\\ &\quad+\sigma_3^{-2}\mathrm{E}\{\|\boldsymbol{\phi}_3(\boldsymbol{I}_n-\boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2\}\\ &= d'_1+d'_2+d'_3 = n-v. \quad (19)\end{aligned}$$

The result (19) is such because, as can be shown,

$$\mathrm{E}\{\|\boldsymbol{\phi}_1(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2\} = \sigma_1^2 d_1', \quad \text{where } d_1' = \mathrm{tr}[\boldsymbol{\phi}_1(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})], \tag{20}$$

$$\mathrm{E}\{\|\boldsymbol{\phi}_2(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2\} = \sigma_2^2 d_2', \quad \text{where } d_2' = \mathrm{tr}[\boldsymbol{\phi}_2(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})], \tag{21}$$

$$\mathrm{E}\{\|\boldsymbol{\phi}_3(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2\} = \sigma_3^2 d_3', \quad \text{where } d_3' = \mathrm{tr}[\boldsymbol{\phi}_3(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})]. \tag{22}$$

With these results it is natural to use as estimators of σ_1^2, σ_2^2 and σ_3^2 the solutions of equations

$$\|\boldsymbol{\phi}_1(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2 = \sigma_1^2 d_1', \tag{23}$$

$$\|\boldsymbol{\phi}_2(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2 = \sigma_2^2 d_2', \tag{24}$$

$$\|\boldsymbol{\phi}_3(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2 = \sigma_3^2 d_3', \tag{25}$$

respectively (as suggested by Nelder, 1968, Section 3). This approach was also advocated by Houtman and Speed (1983, Section 4.5) and applied, e.g., by Caliński and Łacka (2014, p. 959).

For completeness, it will be helpful to note that the equations (23), (24) and (25) imply on account of (19) the equality

$$\begin{aligned}\hat{\sigma}_1^{-2}\|\boldsymbol{\phi}_1(\boldsymbol{I}_n - \boldsymbol{P}_{X(V^{-1})})\boldsymbol{y}\|^2 \; + \; & \hat{\sigma}_2^{-2}\|\boldsymbol{\phi}_2(\boldsymbol{I}_n - \boldsymbol{P}_{X(V^{-1})})\boldsymbol{y}\|^2 \\ & + \hat{\sigma}_3^{-2}\|\boldsymbol{\phi}_3(\boldsymbol{I}_n - \boldsymbol{P}_{X(V^{-1})})\boldsymbol{y}\|^2 \\ & \quad = d_1' + d_2' + d_3' = n - v. \end{aligned} \tag{26}$$

Now, returning to (15), note that, after some algebraic transformations, it can be written equivalently as

$$\begin{aligned}\mathrm{SS}_R \;=\; & \boldsymbol{y}'[\boldsymbol{I}_n - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'](\sigma_1^{-2}\boldsymbol{\phi}_1 + \sigma_2^{-2}\boldsymbol{\phi}_2 + \sigma_3^{-2}\boldsymbol{\phi}_3)[\boldsymbol{I}_n \\ & - \boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}]\boldsymbol{y} \\ = \; & \sigma_1^{-2}\|\boldsymbol{\phi}_1(\boldsymbol{I}_n - \boldsymbol{P}_{X(V^{-1})})\boldsymbol{y}\|^2 + \sigma_2^{-2}\|\boldsymbol{\phi}_2(\boldsymbol{I}_n - \boldsymbol{P}_{X(V^{-1})})\boldsymbol{y}\|^2 \\ & + \sigma_3^{-2}\|\boldsymbol{\phi}_3(\boldsymbol{I}_n - \boldsymbol{P}_{X(V^{-1})})\boldsymbol{y}\|^2. \end{aligned} \tag{27}$$

A comparison of formulae (26) and (27) shows that, if the stratum variances are estimated by solutions of the equations (23), (24) and (25),

$$\begin{aligned}\widehat{\mathrm{SS}}_R \;=\; & \hat{\sigma}_1^{-2}\|\boldsymbol{\phi}_1(\boldsymbol{I}_n - \boldsymbol{P}_{X(V^{-1})})\boldsymbol{y}\|^2 + \hat{\sigma}_2^{-2}\|\boldsymbol{\phi}_2(\boldsymbol{I}_n - \boldsymbol{P}_{X(V^{-1})})\boldsymbol{y}\|^2 \\ & + \hat{\sigma}_3^{-2}\|\boldsymbol{\phi}_3(\boldsymbol{I}_n - \boldsymbol{P}_{X(V^{-1})})\boldsymbol{y}\|^2 = n - v. \end{aligned} \tag{28}$$

On account of (28) the statistic F in (13) is reduced to the form

$$\widehat{F} = \frac{n-v}{v-1}\,\frac{\widehat{\mathrm{SS}}_V}{n-v} = \frac{\widehat{\mathrm{SS}}_V}{v-1}, \tag{29}$$

where $\widehat{\mathrm{SS}}_V$ is as in (14), but with σ_1^2, σ_2^2 and σ_3^2 there replaced by their estimates.

However, the χ^2 distribution of SS_V, indicated in (16), is valid exactly only if the true stratum variances are used in the applied matrix $\boldsymbol{V}^{-1} = \sigma_1^{-2}\boldsymbol{\phi}_1 + \sigma_2^{-2}\boldsymbol{\phi}_2 + \sigma_3^{-2}\boldsymbol{\phi}_3 + \sigma_4^{-2}\boldsymbol{\phi}_4$, resulting from (3). Thus, when using in $\boldsymbol{V}^{-1}$ the estimates of σ_1^2, σ_2^2 and σ_3^2 obtained from (23), (24) and (25), respectively, the distribution (16) can be regarded as approximate only.

4. Some simplifying reformulations

Some reformulation in the methodology presented in Section 3 would simplify the analysis without introducing any changes in its results.

The desirable simplification can be obtained when the matrix $\boldsymbol{V}$ of the form given in (3) is replaced by the matrix

$$\boldsymbol{V}_* = \sigma_1^2\boldsymbol{\phi}_1 + \sigma_2^2\boldsymbol{\phi}_2 + \sigma_3^2(\boldsymbol{I}_n - \boldsymbol{\phi}_1 - \boldsymbol{\phi}_2),$$

i.e., when replacing the inverted matrix $\boldsymbol{V}^{-1}$ by

$$\boldsymbol{V}_*^{-1} = \sigma_1^{-2}\boldsymbol{\phi}_1 + \sigma_2^{-2}\boldsymbol{\phi}_2 + \sigma_3^{-2}(\boldsymbol{I}_n - \boldsymbol{\phi}_1 - \boldsymbol{\phi}_2).$$

The relations between $\boldsymbol{V}$ and $\boldsymbol{V}_*$, and their inverses, are

$$\begin{aligned} \boldsymbol{V} &= \boldsymbol{V}_* + (\sigma_4^2 - \sigma_3^2)n^{-1}\mathbf{1}_n\mathbf{1}_n', \\ \boldsymbol{V}^{-1} &= \boldsymbol{V}_*^{-1} + (\sigma_4^{-2} - \sigma_3^{-2})n^{-1}\mathbf{1}_n\mathbf{1}_n'. \end{aligned} \tag{30}$$

From (30) it follows that

$$(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1} = (\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1} + (\sigma_4^2 - \sigma_3^2)n^{-1}\mathbf{1}_v\mathbf{1}_v'. \tag{31}$$

Applying (31), it can be shown that the BLUE of $\boldsymbol{\tau}_* = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\boldsymbol{\tau}$, i.e., $\hat{\boldsymbol{\tau}}_* = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\hat{\boldsymbol{\tau}} = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{y}$, can equivalently be written as

$$\hat{\boldsymbol{\tau}}_* = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{y}_*, \tag{32}$$

where $\boldsymbol{y}_* = (\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{y}$ with

$\mathrm{E}(\boldsymbol{y}_*) = (\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{X}_1\boldsymbol{\tau} = \boldsymbol{X}_1(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\boldsymbol{\tau} = \boldsymbol{X}_1\boldsymbol{\tau}_*$,

$\mathrm{D}(\boldsymbol{y}_*) = (\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{V}_*(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')$.

The dispersion matrix of $\hat{\boldsymbol{\tau}}_*$ can be presented as

$$\mathrm{D}(\hat{\boldsymbol{\tau}}_*) = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v'). \tag{33}$$

The formulae for SS_V and SS_R, given in (14) and (15), can equivalently be written as

$$\mathrm{SS}_V = \hat{\boldsymbol{\tau}}_*' \boldsymbol{X}_1' \boldsymbol{V}_*^{-1} \boldsymbol{X}_1 \hat{\boldsymbol{\tau}}_* = \boldsymbol{y}_*' \boldsymbol{V}_*^{-1} \boldsymbol{X}_1 (\boldsymbol{X}_1' \boldsymbol{V}_*^{-1} \boldsymbol{X}_1)^{-1} \boldsymbol{X}_1' \boldsymbol{V}_*^{-1} \boldsymbol{y}_*, \tag{34}$$

$$\mathrm{SS}_R = \boldsymbol{y}_*' [\boldsymbol{V}_*^{-1} - \boldsymbol{V}_*^{-1} \boldsymbol{X}_1 (\boldsymbol{X}_1' \boldsymbol{V}_*^{-1} \boldsymbol{X}_1)^{-1} \boldsymbol{X}_1' \boldsymbol{V}_*^{-1}] \boldsymbol{y}_*. \tag{35}$$

The formulae (34) and (35) provide the sum

$$\mathrm{SS}_V + \mathrm{SS}_R = \boldsymbol{y}_*' \boldsymbol{V}_*^{-1} \boldsymbol{y}_* = \mathrm{SS}_T \quad \text{(say)},$$

which can be called the total sum of squares. It can be shown that

$$\mathrm{SS}_T \sim \chi^2(n-1, \delta), \quad \text{with} \quad \delta = \boldsymbol{\tau}_*' \boldsymbol{X}_1' \boldsymbol{V}_*^{-1} \boldsymbol{X}_1 \boldsymbol{\tau}_*. \tag{36}$$

These results can be summarized in the following ANOVA table.

Table 1. The analysis of variance for an experiment in a nested block design with OBS structure

Source of variation	Degrees of freedom	Sum of squares	Expected mean square
Treatments	$v-1$	SS_V	$1+\delta/(v-1)$
Residuals	$n-v$	SS_R	1
Total	$n-1$	SS_T	—

Suppose now that after rejecting the hypothesis H_0 one is interested in testing a hypothesis $H_{0,\mathrm{L}} : \boldsymbol{U}_{\mathrm{L}}'\boldsymbol{\tau} = \mathbf{0}$, where $\boldsymbol{U}_{\mathrm{L}}'\mathbf{1}_v = \mathbf{0}$. Note that this hypothesis can also be written as

$$H_{0,\mathrm{L}} : \boldsymbol{U}_{\mathrm{L}}'\boldsymbol{\tau}_* = \mathbf{0}, \quad \text{where} \quad \boldsymbol{\tau}_* = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\boldsymbol{\tau}, \tag{37}$$

which shows that $H_{0,\mathrm{L}}$ is implied by H_0 given in (10).

Note that the BLUE of $\boldsymbol{U}'_{\mathrm{L}}\boldsymbol{\tau}_*$ is

$$\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_* = \boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}} = \boldsymbol{U}'_{\mathrm{L}}(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{y}_*. \tag{38}$$

Its dispersion matrix is

$$\mathrm{D}(\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_*) = \boldsymbol{U}'_{\mathrm{L}}(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{L}}. \tag{39}$$

Applying Lemma 2.2.6(c) in Rao and Mitra (1971), one can write

$$\boldsymbol{U}'_{\mathrm{L}}(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{L}}[\boldsymbol{U}'_{\mathrm{L}}(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{L}}]^-\boldsymbol{U}'_{\mathrm{L}} = \boldsymbol{U}'_{\mathrm{L}},$$

which gives $\mathrm{D}(\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_*)[\mathrm{D}(\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_*)]^-\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_* = \boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_*$. This shows that the hypothesis $H_{0,\mathrm{L}}$ is consistent. Then (following Theorem 3.2 of Rao, 1971)

$$\begin{aligned}\mathrm{SS}(\boldsymbol{U}_{\mathrm{L}}) &= \hat{\boldsymbol{\tau}}'_*\boldsymbol{U}_{\mathrm{L}}[\mathrm{D}(\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_*)]^-\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_* \\ &= \hat{\boldsymbol{\tau}}'_*\boldsymbol{U}_{\mathrm{L}}[\boldsymbol{U}'_{\mathrm{L}}(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{L}}]^-\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_*, \end{aligned} \tag{40}$$

with the d.f. equal to rank($\boldsymbol{U}_{\mathrm{L}}$), i.e., equal to rank$[\mathrm{D}(\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_*)]$.

Also note that $\boldsymbol{U}_\mathrm{L}[\boldsymbol{U}_\mathrm{L}'(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_\mathrm{L}]^{-}\boldsymbol{U}_\mathrm{L}'$ is invariant for any choice of the involved g-inverse and is of rank equal to the rank of $\boldsymbol{U}_\mathrm{L}$.

Now, following the assumption $\boldsymbol{y} \sim N_n(\boldsymbol{X}_1\boldsymbol{\tau}, \boldsymbol{V})$, adopted in Section 3, one can also assume that

$$\boldsymbol{U}_\mathrm{L}'\hat{\boldsymbol{\tau}}_* \sim N[\boldsymbol{U}_\mathrm{L}'\boldsymbol{\tau}_*, \mathrm{D}(\boldsymbol{U}_\mathrm{L}'\hat{\boldsymbol{\tau}}_*)].$$

With this, applying Theorem 9.2.3 in Rao and Mitra (1971), it can be shown that

$$\mathrm{SS}(\boldsymbol{U}_\mathrm{L}) \sim \chi^2[\mathrm{rank}(\boldsymbol{U}_\mathrm{L}), \delta_L], \quad \text{with} \quad \delta_\mathrm{L} = \boldsymbol{\tau}_*'\boldsymbol{U}_\mathrm{L}[\mathrm{D}(\boldsymbol{U}_\mathrm{L}'\hat{\boldsymbol{\tau}}_*)]^{-}\boldsymbol{U}_\mathrm{L}'\boldsymbol{\tau}_*,$$

this distribution being central if $H_{0,\mathrm{L}}$ is true.

If there are several sets of contrasts for which individual hypothesis testing is of interest, then for each of them the sum of squares presented in (40) can be used accordingly.

In some situations a relevant partition of SS_V may be of interest. For two such sets of contrasts, e.g. $\boldsymbol{U}'_{\mathrm{A}}\boldsymbol{\tau}_*$ and $\boldsymbol{U}'_{\mathrm{B}}\boldsymbol{\tau}_*$, the equality

$$\mathrm{SS}(\boldsymbol{U}_{\mathrm{A}}) + \mathrm{SS}(\boldsymbol{U}_{\mathrm{B}}) = \mathrm{SS}_V \tag{41}$$

holds, for any vector $\hat{\boldsymbol{\tau}}_* = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\hat{\boldsymbol{\tau}}$, if and only if

$$\begin{aligned}&(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{A}}[\boldsymbol{U}'_{\mathrm{A}}(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{A}}]^{-}\boldsymbol{U}'_{\mathrm{A}}\\ &+(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{B}}[\boldsymbol{U}'_{\mathrm{B}}(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{B}}]^{-}\boldsymbol{U}'_{\mathrm{B}} = \boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}'.\end{aligned} \tag{42}$$

This implies that

$$\boldsymbol{U}'_{\mathrm{B}}(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{A}} = \mathbf{O}. \tag{43}$$

These results can be extended for any number of considered sets of contrasts. The condition (43) can then be written as

$$\boldsymbol{U}'_{\mathrm{L}}(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{L}^*} = \mathbf{O} \quad \text{for} \quad \mathrm{L} \neq \mathrm{L}^*. \tag{44}$$

For some classes of designs this condition is reduced to $\boldsymbol{U}'_{\mathrm{L}}\boldsymbol{U}_{\mathrm{L}^*} = \mathbf{O}$.

5. Application with estimated stratum variances

The hypothesis testing procedures presented in Section 4 are fully applicable if the stratum variances σ_1^2, σ_2^2 and σ_3^2 are known. In practical application these variances are usually unknown and have to be estimated.

However, with these estimates the residual sum of squares SS_R is reduced to $n - v$, the corresponding d.f. This leads to relevant reduction of the F statistic (13) to

$$\widehat{F} = \frac{n-v}{v-1} \frac{\widehat{\mathrm{SS}}_V}{n-v} = \frac{\widehat{\mathrm{SS}}_V}{v-1},$$

presented in (29). The estimated treatment (variety) sum of squares, $\widehat{\mathrm{SS}}_V$, can be written as

$$\widehat{\mathrm{SS}}_V = \boldsymbol{y}_*' \hat{\boldsymbol{V}}_*^{-1} \boldsymbol{y}_* - (n - v) \equiv \widehat{\mathrm{SS}}_T - n + v. \tag{45}$$

Because now the matrix $\boldsymbol{V}_*^{-1}$ is replaced by

$$\hat{\boldsymbol{V}}_*^{-1} = \hat{\sigma}_1^{-2}\boldsymbol{\phi}_1 + \hat{\sigma}_2^{-2}\boldsymbol{\phi}_2 + \hat{\sigma}_3^{-2}(\boldsymbol{I}_n - \boldsymbol{\phi}_1 - \boldsymbol{\phi}_2),$$

the estimated total some of squares $\widehat{\mathrm{SS}}_T$, appearing in (45), does not have an exact χ^2 distribution with $n-1$ d.f. That distribution can, however, be considered as an approximation of the real distribution of $\widehat{\mathrm{SS}}_T$. This approximation will be the closer the larger is the number n, i.e., the size of the experiment.

The estimated mean square $\widehat{\mathrm{MS}}_V = \widehat{\mathrm{SS}}_V/(v-1)$ may be treated as having (under H_0) approximately the distribution of $\chi^2(v-1,0)/(v-1)$.

This means that when calculating the relevant P values for testing H_0, or hypotheses implied by H_0, one has to consider them as approximate. The results obtained by Volaufova (2009) seem to suggest that the above ANOVA type F test approximation will in most cases provide reasonably accurate P values. See also the comments in Johnson, Kotz and Balakrishnan (1995, p. 338).

6. Examples

Let the methods be now illustrated using data from three experiments in different NB designs.

All required computations have been accomplished with the use of R (R Core Team, 2017),

Example 1. Brzeskwiniewicz (1994) analyzed data from a $v_{\mathrm{A}} \times v_{\mathrm{B}}$ factorial experiment with $v_{\mathrm{A}} = 3$ doses of nitrogen fertilizer (factor A) and $v_{\mathrm{B}} = 4$ varieties of potato (factor B). The experiment was conducted in an NB design with $\mathcal{D}^*$ based on the incidence matrix

$$\boldsymbol{N} = [\boldsymbol{N}_1 : \boldsymbol{N}_2 : \boldsymbol{N}_3 : \boldsymbol{N}_4 : \boldsymbol{N}_5 : \boldsymbol{N}_6 : \boldsymbol{N}_7 : \boldsymbol{N}_8 : \boldsymbol{N}_9 : \boldsymbol{N}_{10} : \boldsymbol{N}_{11} : \boldsymbol{N}_{12}].$$

Each of the submatrices in N is composed of two columns. This means that there are 24 blocks grouped into 12 superblocks.

Table 2. Experimental observations of the field plot yield of the combinations of the levels of two factors analyzed in Example 1

Block	A	B	Observ.	Block	A	B	Observ.	Block	A	B	Observ.
1	1	1	35.8	9	1	1	38.4	17	2	1	35.3
1	1	3	30.8	9	1	3	30.7	17	2	3	43.8
2	3	1	40.2	10	2	1	28.5	18	3	1	38.0
2	3	3	47.6	10	2	3	37.0	18	3	3	45.4
3	1	1	33.0	11	1	1	36.6	19	2	1	33.6
3	1	4	46.1	11	1	4	44.9	19	2	4	46.0
4	3	1	41.5	12	2	1	30.2	20	3	1	45.0
4	3	4	46.6	12	2	4	50.5	20	3	4	52.0
5	1	2	49.1	13	1	2	48.3	21	2	2	36.0
5	1	3	36.3	13	1	3	35.2	21	2	3	45.5
6	3	2	57.1	14	2	2	41.2	22	3	2	50.6
6	3	3	43.4	14	2	3	47.5	22	3	3	49.0
7	1	2	46.6	15	1	2	49.5	23	2	2	38.5
7	1	4	43.5	15	1	4	44.5	23	2	4	42.3
8	3	2	57.5	16	2	2	46.3	24	3	2	53.3
8	3	4	51.4	16	2	4	42.6	24	3	4	47.0

When analyzing these data, the researcher might be interested in estimating and testing certain sets of treatment parametric functions that can be defined as follows:

$$\left[\left(\boldsymbol{I}_3 - \frac{1}{3}\mathbf{1}_3\mathbf{1}_3'\right)\otimes\frac{1}{4}\mathbf{1}_4'\right]\boldsymbol{\tau} = \boldsymbol{U}_{\mathrm{A}}'\boldsymbol{\tau} \equiv \boldsymbol{U}_{\mathrm{A}}'\boldsymbol{\tau}_*, \tag{46}$$

$$\left[\frac{1}{3}\mathbf{1}_3' \otimes \left(\boldsymbol{I}_4 - \frac{1}{4}\mathbf{1}_4\mathbf{1}_4'\right)\right]\boldsymbol{\tau} = \boldsymbol{U}_{\mathrm{B}}'\boldsymbol{\tau} \equiv \boldsymbol{U}_{\mathrm{B}}'\boldsymbol{\tau}_*, \tag{47}$$

$$\left[\left(\boldsymbol{I}_3 - \frac{1}{3}\mathbf{1}_3\mathbf{1}_3'\right)\otimes\left(\boldsymbol{I}_4 - \frac{1}{4}\mathbf{1}_4\mathbf{1}_4'\right)\right]\boldsymbol{\tau} = \boldsymbol{U}_{\mathrm{AB}}'\boldsymbol{\tau} \equiv \boldsymbol{U}_{\mathrm{AB}}'\boldsymbol{\tau}_* \tag{48}$$

where (46) stands for the main effects of the levels of factor A, (47) stands for the main effects of the levels of factor B, and (48) represents the interaction effects of these two factors.

Table 3. ANOVA of an experiment in an NB design - Example 1

Source of variation	Degrees of freedom	Sum of squares	Mean square
Treatments	11	210.8489	19.1681
Residuals	36	36	1
Total	47	246.8489	—

Table 4. ANOVA for the sets of contrasts considered in Example 1

Source	d.f.	Sum of squares	Mean square	$\hat{F}$	P value
Treatments	11	210.8489	19.1681	19.1681	< 0.0001
A	2	71.3556	35.6778	35.6778	< 0.0001
B	3	97.1209	32.3736	32.3736	< 0.0001
AB	6	42.3724	7.0621	7.0621	< 0.0001
Residuals	36	36	1		
Total	47	246.8489			

The results in Tables 3 and 4 have been obtained with the use of the empirical estimates (i.e., based on $\hat{\sigma}_1^2 = 9.77119$, $\hat{\sigma}_2^2$ = 7.78197 and $\hat{\sigma}_3^2$ = 10.79420)

$$\tilde{\boldsymbol{\tau}} = [36.093,\ 48.159,\ 33.391,\ 44.536,\ 31.836,\ 40.546,\ 43.494,\ 45.288,\ 41.139,\ 54.752,\ 46.247,\ 49.444]'$$

and

$$\tilde{\boldsymbol{\tau}}_* = [-6.818,\ 5.249,\ -9.520,\ 1.626,\ -11.074,\ -2.364,\ 0.584,\ 2.378,\ -1.772,\ 11.842,\ 3.336,\ 6.534]',$$

the former obtainable by the use of formula (8), the latter either from the relation $\tilde{\boldsymbol{\tau}}_* = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\tilde{\boldsymbol{\tau}}$, or directly by formula (32). Entering with $\tilde{\boldsymbol{\tau}}_*$ the formula (34), the estimated sum of squares $\widehat{\mathrm{SS}}_V$ has been obtained. Similarly, using in this way formula (40), the relevant components of $\widehat{\mathrm{SS}}_V$ have been obtained. Evidently, as it follows from (28), the estimated residual sum of squares $\widehat{\mathrm{SS}}_R$ is reduced to $n - v$, its d.f. As to the term "empirical estimates", it has been taken from Rao and Kleffe (1988, p. 274).

Example 2. Caliński and Łacka (2014) analyzed data from a plant protection experiment. The experiment was carried out in laboratory conditions, in a growth chamber. Its aim was to evaluate the efficiency of 4 chemical substances (levels of factor B) applied, in 3 concentrations, low, mid and high (levels of factor A), to reduce plant damage caused by slugs *Arion lusitanicus*. In the conducted experiment, discs of Chinese cabbage leaves were treated with relevant solutions of the studied chemical compounds, and the size of their damage made by slugs, given in percents of their surfaces, was observed. Each box, as an experimental unit, contained 3 such discs and one *A. lusitanicus* slug placed inside. One camera covering $k = 2$ boxes was considered as forming a block of the design. During the experiment, $b_0 = 3$ cameras were working simultaneously. Each series of watching with the use of these 3 cameras was considered as one superblock of the design. In total, the experiment was composed of $a = 8$ such series, giving $b = 24$. Thus, in the experiment there were $n = 48$ experimental units, allowing each of the $v = 12$ treatments to be replicated $r = 4$ times.

Table 5. Observed damage to Chinese cabbage leaves' discs (in percents of their surfaces) at 3 concentrations (A) of 4 studied chemicals (B).

Block	A	B	Observ.	Block	A	B	Observ.	Block	A	B	Observ.
1	1	1	62.75	9	3	2	14.20	17	2	1	2.12
1	1	3	7.90	9	3	3	13.50	17	2	4	1.25
2	2	1	1.90	10	1	2	74.00	18	3	1	13.00
2	2	3	1.70	10	1	4	4.90	18	3	4	12.80
3	3	1	12.70	11	2	2	36.25	19	1	2	77.00
3	3	3	13.00	11	2	4	0.80	19	1	3	9.00
4	1	1	64.55	12	3	2	13.90	20	2	2	38.20
4	1	4	5.50	12	3	4	12.02	20	2	3	2.90
5	2	1	2.78	13	1	1	63.45	21	3	2	15.20
5	2	4	1.35	13	1	3	7.90	21	3	3	13.10
6	3	1	13.70	14	2	1	3.20	22	1	2	75.25
6	3	4	13.00	14	2	3	1.80	22	1	4	5.10
7	1	2	74.95	15	3	1	14.20	23	2	2	38.00
7	1	3	8.40	15	3	3	14.00	23	2	4	1.40
8	2	2	37.55	16	1	1	65.25	24	3	2	14.30
8	2	3	2.40	16	1	4	5.70	24	3	4	13.60

When analyzing these data, the researcher was particularly interested in estimating and testing certain set of contrasts. They can be presented as some basic contrasts $\{\boldsymbol{c}_i'\boldsymbol{\tau} \equiv \boldsymbol{c}_i'\boldsymbol{\tau}_*, \;\; i = 1, 2, ..., 11\}$ determined by the following vectors:

$$\begin{aligned}
\boldsymbol{c}_1 &= \mathbf{1}_3 \otimes [-1,\ -1,\ 1,\ 1]'/\sqrt{3},\\
\boldsymbol{c}_2 &= [-2,\ 1,\ 1]' \otimes [-1,\ -1,\ 1,\ 1]'/\sqrt{6},\\
\boldsymbol{c}_3 &= [0,\ -1,\ 1]' \otimes [-1,\ -1,\ 1,\ 1]'/\sqrt{2},\\
\boldsymbol{c}_4 &= \mathbf{1}_3 \otimes [-1,\ 1,\ 0,\ 0]'\sqrt{2}/\sqrt{3},\\
\boldsymbol{c}_5 &= \mathbf{1}_3 \otimes [0,\ 0,\ -1,\ 1]'\sqrt{2}/\sqrt{3},\\
\boldsymbol{c}_6 &= [-2,\ 1,\ 1]' \otimes [-1,\ 1,\ 0,\ 0]'/\sqrt{3},\\
\boldsymbol{c}_7 &= [-2,\ 1,\ 1]' \otimes [0,\ 0,\ -1,\ 1]'/\sqrt{3},\\
\boldsymbol{c}_8 &= [0,\ -1,\ 1]' \otimes [-1,\ 1,\ 0,\ 0]',\\
\boldsymbol{c}_9 &= [0,\ -1,\ 1]' \otimes [0,\ 0,\ -1,\ 1]',\\
\boldsymbol{c}_{10} &= [-2,\ 1,\ 1]' \otimes \mathbf{1}_4/\sqrt{6},\\
\boldsymbol{c}_{11} &= [0,\ -1,\ 1]' \otimes \mathbf{1}_4/\sqrt{2}.
\end{aligned}$$

Table 6. ANOVA of an experiment in an NB design - Example 2

Source of variation	Degrees of freedom	Sum of squares	Mean square
Treatments	11	103246.2	9386.018
Residuals	36	36	1
Total	47	103282.2	—

Table 7. ANOVA for the contrasts considered in Example 2

Source	d.f.	Sum of squares	Mean square	$\hat{F}$	P value
Treatments	11	103246.20	9386.02	9386.02	< 0.0001
$\boldsymbol{c}_1'\boldsymbol{\tau}$	1	38874.34	38874.34	38874.34	< 0.0001
$\boldsymbol{c}_2'\boldsymbol{\tau}$	1	32927.98	32927.98	32927.98	< 0.0001
$\boldsymbol{c}_3'\boldsymbol{\tau}$	1	2669.24	2669.24	2669.24	< 0.0001
$\boldsymbol{c}_4'\boldsymbol{\tau}$	1	3690.55	3690.55	3690.55	< 0.0001
$\boldsymbol{c}_5'\boldsymbol{\tau}$	1	28.52	28.52	28.52	< 0.0001
$\boldsymbol{c}_6'\boldsymbol{\tau}$	1	214.89	214.89	214.89	< 0.0001
$\boldsymbol{c}_7'\boldsymbol{\tau}$	1	26.57	26.57	26.57	< 0.0001
$\boldsymbol{c}_8'\boldsymbol{\tau}$	1	4125.63	4125.63	4125.63	< 0.0001
$\boldsymbol{c}_9'\boldsymbol{\tau}$	1	0.69	0.69	0.69	$= 0.4107$
$\boldsymbol{c}_{10}'\boldsymbol{\tau}$	1	20526.73	20526.73	20526.73	< 0.0001
$\boldsymbol{c}_{11}'\boldsymbol{\tau}$	1	161.06	161.06	161.06	< 0.0001
Residuals	36	36	1		
Total	47	103282.20			

The results presented in Tables 6 and 7 have been obtained with the use of the empirical estimates (i.e., based on $\hat{\sigma}_1^2 = 0.23019$, $\hat{\sigma}_2^2 = 0.35245$ and $\hat{\sigma}_3^2 = 1.53393$)

$$\begin{aligned}\tilde{\boldsymbol{\tau}} = [&64.037,\ 75.263,\ \ 8.275,\ \ 5.325,\ \ 2.564,\ 37.436,\\ &2.107,\ \ 1.293,\ 13.412,\ 14.388,\ 13.315,\ 12.940]'\end{aligned}$$

and

$$\begin{aligned}\tilde{\boldsymbol{\tau}}_* = \ [&43.174,\ \ 54.400,\ -12.588,\ -15.538,\ -18.299,\ \ 16.573,\\ &-18.756,\ -19.570,\ \ -7.451,\ \ -6.475,\ \ -7.548,\ -7.923]',\end{aligned}$$

following the same approach as that applied in Example 1.

Example 3. Ceranka (1983) analyzed data from a plant-breeding field experiment with 25 breeding strains and 2 standard varieties of sunflower compared in an NB design based on the incidence matrix $\boldsymbol{N}^*$ of the type

$$\boldsymbol{N}^* = \begin{bmatrix} \boldsymbol{N} \\ \mathbf{1}_s \mathbf{1}_b' \end{bmatrix}.$$

It has $b = 30$ blocks, each of size $k = 7$, grouped into $a = 6$ superblocks, each of size $n_0 = 35$. Note that the design by which the 27 treatments are arranged into 6 superblocks is here based on the 27×6 incidence matrix

$$\boldsymbol{M} = \begin{bmatrix} \mathbf{1}_{25} & \mathbf{1}_{25} & \mathbf{1}_{25} & \mathbf{1}_{25} & \mathbf{1}_{25} & \mathbf{1}_{25} \\ 5\mathbf{1}_2 & 5\mathbf{1}_2 & 5\mathbf{1}_2 & 5\mathbf{1}_2 & 5\mathbf{1}_2 & 5\mathbf{1}_2 \end{bmatrix}.$$

Table 8. ANOVA of an experiment in an NB design - Example 3

Source	d.f.	Sum of squares	Mean square	$\hat{F}$	P value
Treatments	26	93.2401	3.5862	3.5862	< 0.0001
Residuals	183	183	1		
Total	209	276.2401			

The results presented in Table 8 have been obtained with the use of the empirical estimates (i.e., based on $\hat{\sigma}_1^2 = 0.91939$, $\hat{\sigma}_2^2 = 2.03387$ and $\hat{\sigma}_3^2 = 201.50110$)

$$\begin{aligned}\tilde{\boldsymbol{\tau}} = \; & [15.552,\ 15.808,\ 15.099,\ 15.627,\ 16.347,\ 15.725,\ 15.426,\ 15.747,\ 16.025,\\ & 15.290,\ 16.418,\ 15.454,\ 15.430,\ 15.664,\ 15.591,\ 15.811,\ 15.300,\ 16.564,\\ & 15.473,\ 17.628,\ 14.937,\ 15.211,\ 14.618,\ 14.678,\ 14.326,\ 14.690,\ 15.780]',\end{aligned}$$

$$\begin{aligned}\tilde{\boldsymbol{\tau}}_* = \; & [0.063,\ 0.319,\ -0.390,\ 0.138,\ 0.858,\ 0.237,\ -0.062,\ 0.259,\ 0.537,\\ & -0.198,\ 0.929,\ -0.034,\ -0.058,\ 0.176,\ 0.102,\ 0.322,\ -0.189,\ 1.076,\\ & -0.015,\ 2.140,\ -0.552,\ -0.277,\ -0.871,\ -0.811,\ -1.163,\ -0.799,\ 0.291]',\end{aligned}$$

following the same approach as that applied in Example 1.

7. Concluding remarks

- The discovered results concerning the proposed approach to ANOVA for experiments with orthogonal block structure seem to be useful.
- The analytical procedures presented in Section 4 simplify the computations very much.
- The main advantage of the proposed approach is the fact that the ANOVA results are obtainable directly, not by performing first some partial analyses, under relevant stratum submodels, and then combining their results.
- It is expected that this paper will be followed by other articles, devoted to different classes of designs inducing the OBS property.

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